Missile Stability Using Finite Elements—an Unconstrained Variational Approach

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The stability behavior of a flexible missile idealized as a free-free beam is studied using the finite-element technique. The structure is assumed to be under a constant thrust which, in turn, is subjected to a directional control device. A concentrated mass is included to model a piece of heavy machinery. The solution formulation – finite element in conjunction with unconstrained variational principles – is shown here to be general, simple to use, and effective to overcome the difficulties arising from nonconservative forces, concentrated mass, and feedback control features. As a basis of this approach, an unconstrained variational statement and the associated adjoint problems are introduced. Numerical results from this study reveal that 1) for a free-free beam under a constant thrust, there exists a nonzero mode which seems to have escaped previous investigators; 2) since this newly realized mode is the lowest nonzero mode and is divergent in nature, a missile structure is unstable without feedback control; and 3) depending on the amount and location of a concentrated mass, it can improve the stability behavior of a missile.

I. Introduction

To assure the precise location and direction of a space rocket or a missile at any moment during the flight, its vibration and stability behavior are obviously of paramount importance. The stability of deformable missiles, idealized as free-free beams, has been investigated by several writers.[1-7] Beal[1] considered a uniform beam under constant and pulsating thrusts with a simple feedback directional control. Kaliski and Woroszyl,[3] Kacprzyniski and Kaliski,[4] and Solarz[5] investigated more complicated problems by including such effects as the aerodynamic forces, material damping, and the cross-sectional variations along the missile's length. More recently, Kalisnikov and Il'in[6] extended Beal's results to include material damping and more general capabilities of the directional control of thrusts. Matsumoto and Mote[7] have further included the effect of time delay in the directional control feedback. One of the special features of their paper is that the finite-element technique was used in conjunction with the τ-decomposition method.[8]

Since the first appearance of the pioneer work by Nikolai,[9] there has been rapid progress in the theory of nonconservative stability in the last two decades. A text by Bolotin[10] deals exclusively with such problems. A review paper by Herrmann[11] contains a comprehensive bibliography of the work on the subject performed prior to 1967.

Mote[12] and Barsoum[13] were the first to apply the finite-element method to nonconservative stability problems. Both writers employed the extended Hamilton's principle as their basis of approximations. Later Kikuchi[14] presented a Galerkin-finite-element approach to general nonself adjoint problems. In the meantime, Wu[15] has successfully solved nonconservative stability problems using finite elements in conjunction with adjoint variational statements. It has been found subsequently that the general linear, nonconservative stability problems can be solved most efficiently using finite elements formulated with unconstrained adjoint variational statements.[16-19] One of the purposes of the present paper is to extend such formulations to the missile stability problem with concentrated mass and a thrusting force with directional control.

When directional control is not included, the present numerical results reveal the following: 1) associated with a uniform beam under a constant thrust, there exists a nonzero eigenvalue which represents a divergent instability, and 2) the amount and location of a concentrated mass can be chosen to improve the stability behavior of a missile structure.

First the mathematical modeling of the problem is described in the Sec. II. An unconstrained variational statement and the associated adjoint problems are introduced in Sec. III. The finite-element formulations are then given. In Sec. V, the stability criteria is recapitulated together with a discussion on the rigid body modes of free-free beams. Finally the numerical results are presented and their significance discussed.

II. Mathematical Modeling of a Flexible Missile

The sketch of a typical missile (or rocket) structure is shown in Fig. 1a. First the structure is assumed to be slender enough that a good evaluation on its stability behavior can be obtained using the simple beam theory. The cross-sectional variation can be easily included in the analysis since the finite-element method will be used. A piece of heavy machinery is modeled by a concentrated mass at any desired location.

The jet propulsion is represented by a compressive force, considered constant in the present study, at the tail end. The direction of this thrust is controlled by a feedback sensor located at a predetermined position. Aerodynamic forces are neglected in this study.

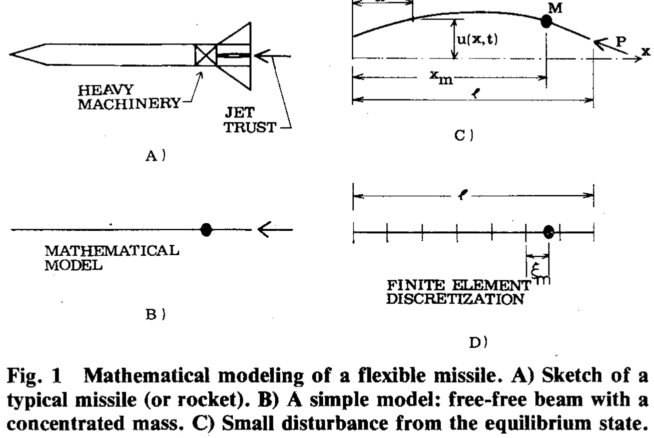

Fig. 1 Mathematical modeling of a flexible missile. A) Sketch of a typical missile (or rocket). B) A simple model: free-free beam with a concentrated mass. C) Small disturbance from the equilibrium state. D) Finite element discretization.

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In any particular moment during its flight, the missile structure can be considered in a state of dynamic equilibrium. In this state, the structure can be assumed to be rectilinear in form, referred to as the undisturbed axis, without loss of generality. The lateral stability of the structure is then characterized by the behavior of a small disturbance from this axis and perpendicular to it (Fig. 1b).

Let $u(x,t)$ denote such a disturbance. The differential equation and boundary conditions can be written as the following

$$D.E.\quad (EIu'')'' + (P^*(x)u')' + \rho A\ddot{u} + \frac{M}{\ell}\{\ddot{u}(x) + \frac{P}{\rho A\ell + M}u'(x_m)\}\delta(x-x_m) = 0 \tag{1}$$

$$B.C.\quad u''(0) = u'''(0) = 0 \tag{2a}$$

$$u''(\ell) = u'''(\ell) - \frac{Pk_\theta}{EI}u'(x_\theta) = 0 \tag{2b}$$

where

$$P^*(x) = \left[\frac{\rho A x}{\rho A\ell + M}\right]P \text{ for } 0 \le x < x_m \tag{3a}$$

$$= \left[\frac{\rho A x + M}{\rho A\ell + M}\right]P \text{ for } x_m \le x \le 1 \tag{3b}$$

Other parameters are defined as follows:

A, I = area, second moment of the cross section
ℓ = missile length
ρ, E = mass per unit length, Young's modulus
P = thrusting force
M = concentrated mass
K_θ = thrust directional control constant

A prime ($'$) denotes a differentiation with respect to the spatial coordinate x, measured along the missile's length, and a dot ($\cdot$), differentiation with respect to time t. The parameters x_m and x_θ mark the locations of the concentrated mass and the feedback sensor.

It will be convenient to deal with nondimensionalized forms of the equations. To this end, the following nondimensional parameters are introduced.

$$x_{(\text{new})} = x_{(\text{old})}/\ell, \quad u_{(\text{new})} = u_{\text{old}}/\ell \tag{4a}$$

$$t_{(\text{new})} = t_{(\text{old})}/c, \quad m = M/\rho A\ell \tag{4b}$$

$$Q = P\ell^2/EI, \quad Q_1 = Q/(1+m) \tag{4c}$$

where $c = (\rho A\ell^4/EI)^{1/2}$ has the dimension of time.

In terms of the new parameters of Eqs. (4), the nondimensional forms of Eqs. (1) and (2) become

$$u'''' + (Q_1 x u')' + mQ_1 u'' H(x-x_m) + \lambda^2 u'' + \{m\lambda^2 u(x_m) + mQ_1 u'(x_m)\}\delta(x-x_m) = 0 \tag{5}$$

$$u''(0) = u'''(0) = 0$$

$$u''(1) = u'''(1) - QK_\theta u'(x_\theta) = 0 \tag{6}$$

where $H(x)$ and $\delta(x)$ are the Heaviside step function and the Dirac delta function, respectively.

Equations (5) and (6) are valid for constant cross sections only. For the purpose of the present study, we shall limit outselves to these equations. The extension of the following formulations to varying corss-sectional properties is quite straightforward.

It should also be noted that in Eqs. (5) and (6), the usual assumption that

$$u(x,t) = u(x)e^{\lambda t} \tag{7}$$

has been adapted and there should be no confusion caused by the simplified notation of using the same letter u for two different functions $u(x,t)$ and $u(x)$.

III. An Unconstrained Variational Principle

As the basis of the approximate solution, the following variational statement is introduced

$$\delta I = 0 \tag{8}$$

where

$$I = \int_0^1 [u''v'' - Q_1 x u'v' - mQ_1 u'v' H(x-x_m) + \lambda^2 uv]\,dx + Qu'(1)\,v(1) + QK_\theta u'(x_\theta)v(1) + m\lambda^2 u(x_m)v(x_m) \tag{9}$$

Carrying out the variation of Eq. (8) using integration by parts, it is not difficult to arrive at the following

$$\begin{aligned}\delta I = &\int_0^1 [u'''' + (Q\,xu') + mQ_1 u' H(x-x_m) + \lambda^2 u + (m\lambda^2 u(x_m) + mQ_1 u'(x_m))\ \delta(x-x_m)]\ \delta v\,dx \\ &+ u''(1)\delta v'(1) - [u'''(1) - QK_\theta u'(x_\theta)]\delta v(1) \\ &- u''(0)\delta v'(0) + u'''(0)\delta v(0) \\ &+ \int_0^1 [v'''' + (Q_1 x v') + mQ_1 v' H(x-x_m) + \lambda^2 v - QK_\theta v(1)\delta'((x-x_\theta) + (m\lambda^2 v(x_m) + mQ_1 v'(x_m)\delta(x-x_m)\delta u\,dx \\ &+ [v''(1) + Qv(1)]\ \delta u'(1) - [v'''(1) + Qv'(1)]\delta u(1) \\ &- v''(0)\delta u'(0) + v'''(0)\ \delta u(0)\end{aligned} \tag{10}$$

In Eq. (10),

$$\delta'(x-x_\theta) = \frac{d}{dx}\delta(x-x_\theta) \tag{11}$$

There should be no confusion between the variation symbol and the Dirac function. Physically, $\delta(x)$ represents a concentrated force and $\delta'(x)$ may be considered as a couple, or a double force, well-known in the theory of elasticity. The mathematical justification of the use of the δ function and all of its derivatives was given by Schwartz.[20]

Now since the following variations are allowed to be completely arbitrary, i.e.,

δu, δv in interval $0 < x < 1$

$\delta u(1)$, $\delta u'(1)$, $\delta v'(1)$ $\delta v'(1)$

$\delta u(0)$, $\delta u'(0)$, $\delta v(0)$, $\delta v'(0)$

The necessary and sufficient condition for Eq. (8) is the following two boundary value problems.

The original problem,

$$D.E.\quad u'''' + (Q\,xu')' + mQ\,u'H(x-x_m) + \lambda^2 u + (m\lambda^2 u(x_m) + mQ_1 u'(x_m))\ \delta(x-x_m) = 0 \tag{12a}$$

$$\begin{aligned}B.C.\quad & u''(1) = 0 \\ & -[u'''(1) - QK_\theta u'(x_\theta)] = 0 \\ & -u''(0) = 0 \\ & u'''(0) = 0\end{aligned} \tag{12b}$$

and the adjoint problem,

$$\text{D.E.}\quad v'''' + (Q_1 x v')' + mQ_1 v' H(x-x_m) + \lambda^2 v + (m\lambda^2 v(x_m) + mQ_1 v'(x_m))\,\delta(x-x_m) - QK_\theta v(1)\,\delta'(x-x_m) = 0 \tag{13a}$$

$$\text{B.C.}\quad v''(1) + Qv(1) = 0$$
$$-[v'''(1) + Qv'(1)] = 0$$
$$-v''(0) = 0$$
$$v'''(0) = 0 \tag{13b}$$

IV. Finite-Element Formulations

To derive the finite-element matrix eigenvalue equations, one begins with the variational statement of Eq. (8).

$$\delta I = 0 \tag{14a}$$

$$I = \int_0^1 (u''v'' - Q_1 x u'v' - mQ_1 u'v' H(x-x_m) + \lambda^2 uv)\,dx + Qu'(1)\,v(1) + QK_\theta u'(x)v(1) + m\lambda^2 u(x_m)v(x_m) \tag{14b}$$

Carrying out the variation of Eq. (14) one arrives at

$$\delta I = \delta I_1 + \delta I_2 = 0 \tag{15}$$

where, symbolically,

$$\delta I_1 = \int_0^1 (u'\delta v'' - Q_1 x u'\delta v' - mQ\,u'\delta v' + \lambda^2 u\delta v)\,dx + Qu'(1)\delta v(1) + QK_\theta u'(x_\theta)\delta v(1) + m\lambda^2 u(x_m)\delta v(x_m) \tag{16a}$$

$$\delta I_2 = \int_0^1 (v''\delta u'' - Q_1 x v'\delta u' - mQ_1 v'\delta u' + \lambda\, v\delta u)\,dx + Qv(1)\delta u'(1) + QK_\theta v(1)\delta u'(x_\theta) + m\lambda^2 v(x_m)\delta u(x_m) \tag{16b}$$

Since δI_1 involves only δv, and δI_2, only δu, and since δu and δv are quite independent of each other, it is only necessary to consider either $\delta I_1 = 0$ or $\delta I_2 = 0$ for the eigenvalues. It might be of interest to point out that the final matrix equation from $\delta I_1 = 0$ is simply the transpose of that from $\delta I_2 = 0$. This is one way to be convinced that the set of eigenvalues of the adjoint problem is the same as the original problem.

Now, the slender beam structure is divided into L segments (elements) as shown in Fig. 1c. In the discretized structure, one has

$$\delta I_1 = 0 \tag{17a}$$

$$= \sum_{i=1}^{L} \int_0^1 \{L^3 u^{(i)''}\delta v^{(i)''} - Q_1(\xi^{(i)} + i - 1)u^{(i)'}\delta v^{(i)'} + \frac{\lambda^2}{L} u^{(i)}\delta v^{(i)}\}\,d\xi$$
$$- mQ_1 L \int_0^1 u^{(i_m)'}\delta v^{(i_m)'}\,\mathrm{O}(\xi - \xi_m)\,d\xi - \sum_{i=i_m+1}^{L} mQ_1 L \int_0^1 u^{(i)'}\delta v^{(i)'}\,d\xi$$
$$+ QLu^{(L)'}(1)\delta v^{(L)}(1) + QK_\theta L u^{(i_\theta)'}(\xi_\theta)\delta v^{(L)}(1) + m\lambda^2 u^{(i_m)}(\xi_m)\delta v^{(i_m)}(\xi_m) \tag{17b}$$

In obtaining Eqs. (17), a change of independent variable has been implemented such that

$$\xi^{(i)} = Lx - i + 1 \tag{18}$$

where the superscript (i) denotes "the ith element."

Thus

$$\xi^{(i_\theta)} = L_{x_\theta} - i_\theta + 1 \tag{19a}$$

$$\xi^{(i_m)} = L\,x_m - i_m + 1 \tag{19b}$$

where x_m and x_θ designate the locations of the concentrated mass and the feedback control.

Now, let

$$u^{(i)}(\zeta) = \boldsymbol{a}^T(\zeta)\,U^{(i)} \tag{20a}$$

$$v^{(i)}(\zeta) = \boldsymbol{a}^T(\zeta)\,V^{(i)} \tag{20b}$$

where

$$U^{(i)T} = \{U_1^{(i)}, U_2^{(i)}, U_3^{(i)}, U_4^{(i)}\} \tag{21a}$$

$$V^{(i)T} = \{V_1^{(i)}, V_2^{(i)}, V_3^{(i)}, V_4^{(i)}\} \tag{21b}$$

are the generalized coordinate vectors and

$$\boldsymbol{a}^T(\zeta) = \{1 - 3\zeta^2 + 2\zeta^3,\ \zeta - 2\zeta^2 + \zeta^3,\ 3\zeta^2 - 2\zeta^3,\ -\zeta^2 + \zeta^3\} \tag{22}$$

is the shape function vector (see Ref. 15).

Using Eqs. (20) in Eq. (17) one obtains

$$\delta I_1 = \sum_{i=1}^{L} \boldsymbol{U}^{(i)T}[\boldsymbol{R}^{(i)} + \lambda^2 \boldsymbol{p}^{(i)}]\,\delta \boldsymbol{V}^{(i)} + \boldsymbol{U}^{(L)T}\boldsymbol{E}\delta\boldsymbol{V}^{(L)} + \boldsymbol{U}^{(i_\theta)T}\boldsymbol{F}\delta\boldsymbol{V}^{(L)} + \lambda^2 \boldsymbol{U}^{(i_m)T}\boldsymbol{G}\delta\boldsymbol{V}^{(i_m)} = 0 \tag{23}$$

where

$$\boldsymbol{R}^{(i)} = \begin{cases} L^3\boldsymbol{C} - Q_1(i-1)\boldsymbol{B} - Q_1\boldsymbol{D}, & i < i_m \\ L^3\boldsymbol{C} - Q_1(i-1)\boldsymbol{B} - Q_1\boldsymbol{D} - mQ_1 L\boldsymbol{H}, & i = i_m \\ L^3\boldsymbol{C} - Q_1(i-1)\boldsymbol{B} - Q_1\boldsymbol{D} - mQ_1 L\boldsymbol{B}, & i > i_m \end{cases} \tag{24a}$$

$$\boldsymbol{P}^{(i)} = \frac{1}{L}\boldsymbol{A} \tag{24b}$$

$$\boldsymbol{E} = QL\boldsymbol{E}_1 \tag{24c}$$

$$\boldsymbol{F} = -QK_\theta \boldsymbol{F}_1 \tag{24d}$$

$$\boldsymbol{G} = m\boldsymbol{G}_1 \tag{24e}$$

$$\boldsymbol{A} = \int_0^1 \boldsymbol{a}(\zeta)\boldsymbol{a}^T(\zeta)\,d\zeta, \qquad \boldsymbol{B} = \int_0^1 \boldsymbol{a}'(\zeta)\boldsymbol{a}'^T(\zeta)\,d\zeta \tag{25a}$$

$$\boldsymbol{C} = \int_0^1 \boldsymbol{a}''(\zeta)\boldsymbol{a}''^T(\zeta)\,d\zeta, \qquad \boldsymbol{D} = \int_0^1 \zeta\boldsymbol{a}'(\zeta)\boldsymbol{a}'^T(\zeta)\,d\zeta \tag{25b}$$

$$\boldsymbol{E}_1 = \boldsymbol{a}'(1)\boldsymbol{a}^T(1), \qquad \boldsymbol{F}_1 = \boldsymbol{a}'(\zeta_\theta)\boldsymbol{a}^T(1) \tag{25c}$$

$$\boldsymbol{G}_1 = \boldsymbol{a}(\zeta_m)\boldsymbol{a}^T(\zeta_m), \boldsymbol{H} = \int_{\zeta m}^1 \boldsymbol{a}'(\zeta)\boldsymbol{a}'^T(\zeta)\,d\zeta \tag{25d}$$

Using the continuity requirement that

$$\left.\begin{aligned} U_3^{(i-1)} &= U_1^{(i)}, & V_3^{(i-1)} &= V_1^{(i)} \\ U_4^{(i-1)} &= U_2^{(i)}, & V_4^{(i-1)} &= V_2^{(i)} \end{aligned}\right\}\ i = 2,3,\ldots,L \tag{26a, 26b}$$

One can assemble Eq. (23) into the global matrix equation

$$\delta I_1 = U^T (R + \lambda^2 P) \delta V = 0 \tag{27}$$

where

$$U^T = \{ U_1^{(1)}, U_2^{(1)}, U_1^{(2)}, U_2^{(2)}, \ldots, U_1^{(L)}, U_2^{(L)}, U_3^{(L)}, U_4^{(L)} \} \tag{28a}$$

$$V^T = \{ V_1^{(1)}, V_2^{(1)}, V_1^{(2)}, V_2^{(2)}, \ldots, V_1^{(L)}, V_2^{(L)}, V_3^{(L)}, V_4^{(L)} \} \tag{28b}$$

The construction of R and P will be conveniently described in matrix partition concept.

Let us rewrite Eqs. (28) as

$$U^T = \{ U^{(1)}, U^{(2)}, \quad U^{(L)}, U^{(L+1)} \} \tag{29}$$

where

$$U^{(i)} = \{ U_1^{(i)}, U_2^{(i)} \}, \ i = 1,2,\cdots L \tag{30a}$$

$$U^{(L+1)} = \{ U_3^{(L)}, U_4^{(L)} \} \tag{30b}$$

and similarly for V^T. Then matrices R and P can be considered as $(L+1)$ by $(L+1)$ with the elements themselves being matrices of 2×2.

Also, in Eqs. (24) one writes

$$R^{(i)} = \begin{bmatrix} R_{11}^{(i)} & R_{12}^{(i)} \\ R_{21}^{(i)} & R_{22}^{(i)} \end{bmatrix} \tag{31}$$

with

$$R_{11}^{(i)} = \begin{bmatrix} R_{11}^{(i)} & R_{12}^{(i)} \\ R_{21}^{(i)} & R_{22}^{(i)} \end{bmatrix}, \quad R_{12}^{(i)} = \begin{bmatrix} R_{13}^{(i)} & R_{14}^{(i)} \\ R_{23}^{(i)} & R_{24}^{(i)} \end{bmatrix} \tag{32a}$$

$$R_{21}^{(i)} = \begin{bmatrix} R_{31}^{(i)} & R_{32}^{(i)} \\ R_{41}^{(i)} & R_{42}^{(i)} \end{bmatrix}, \quad R_{22}^{(i)} = \begin{bmatrix} R_{33}^{(i)} & R_{34}^{(i)} \\ R_{43}^{(i)} & R_{44}^{(i)} \end{bmatrix} \tag{32b}$$

and similarly for other matrices in Eq. (24).

In consideration of Eq. (23), one writes for the matrices R and P of Eq. (27) as

$$R = \bar{R} + R^E + R^F \tag{33a}$$

$$P = \bar{P} + P^G \tag{33b}$$

Then it is easily shown that for the elements in $\bar{R}$:

$$\bar{R}_{11} = R_{11}^{(1)} \tag{34a}$$

$$\bar{R}_{(L+1)(L+1)} = R_{22}^{(L)} \tag{34b}$$

$$\bar{R}_{ii} = R_{11}^{(i)} + R_{22}^{(i-1)} \tag{34c}$$

$$\bar{R}_{(i-1)i} = R_{12}^{(i)} \qquad i = 2,3,\ldots,L \tag{34d}$$

$$\bar{R}_{i(i-1)} = R_{21}^{(i)} \tag{34e}$$

All other elements of $\bar{R}$ undefined in Eqs. (34) are zero. The matrix $\bar{P}$ is constructed in a parallel manner.

Also easily observed are the elements for R^E, R^F, and P^G:

$$R_{LL}^E = E_{11}, R_{i_\theta L}^F = F_{11}, P_{i_m i_m}^G = G_{11} \tag{35a}$$

$$R_{L(L+1)}^E = E_{12}, R_{i_\theta (L+1)}^F = F_{12}, P_{i_m (i_m+1)}^G = G_{12} \tag{35b}$$

$$R_{(L+1)L}^E = E_{21} R_{(i_\theta+1)L} = F_{21}, P_{(i_m+1)i_m}^G = G_{21} \tag{35c}$$

$$R_{(L+1)(L+1)}^E = E_{22}, R_{(i_\theta+1)(L+1)} = F_{22}, P_{(i_m+1)(i_m+1)}^G = G \tag{35d}$$

It is thus clear that R and P are essentially banded matrices. The scattered nonzero elements are only those defined in Eqs. (35).

V. Stability Crtiteria and Rigid Body Modes

As mentioned in Sec. II, the usual assumption has been made that

$$u(x,t) = u(x) e^{\lambda t} \tag{36}$$

The parameter s is a complex number in general. Thus, one writes

$$\lambda = \lambda_R + i\lambda_I \tag{37}$$

where $i = \sqrt{-1}$, and λ_R and λ_I are both real numbers.

Where λ_R is negative or zero, the disturbance $u(x,t)$ will decrease with the time t or remain finite. Thus the structure is considered stable.

When λ_R is nonzero and positive, the disturbance will grow with time. This is the case of instability. In agreement with usual terminology, divergence instability is characterized by $\lambda_I = 0$, whereas flutter (or oscillatory) instability, by $\lambda_I \neq 0$.

In the special case where both λ_R and λ_I are zero, the solution represents a rigid body motion. Within the context of the small oscillation theory, based on which the governing equations are written, the term "rigid body motion" obviously has restricted meaning. It implies an arbitrary, (none the less) small motion. Such a motion, be it translation or rotation, is necessarily small and is in such a manner so that the state of (dynamic or static) equilibrium is not altered. Or, what is the same, the original set of equations is still valid. With this understanding, the solutions associated with rigid body modes should therefore be considered stable modes. This point is emphasized due to some unthinkable arguments made on the "unstable" nature of the rigid body rotation by several writers.

To understand fully the nature of the rigid body modes of a flexible missile, it may be helpful to review some simple cases. The lateral oscillation of a free-free beam (Fig. 2a) can be described by the following differential equation and boundary conditions:

$$D.E. \qquad U'''' + \lambda^2 U = 0 \tag{38a}$$

$$B.C. \qquad U''(0) = U'''(0) = 0 \tag{38b}$$

$$U''(1) = U'''(1) = 0 \tag{38c}$$

It is a simple matter to show that λ^2 has two zero solutions corresponding to two rigid body modes: u = constant; u' = constant.

Another example for which there exists the same two zero eigenvalue solutions is a beam subjected to compressive follower force Q at both ends (Fig. 2b). For this problem, one has

$$D.E. \qquad U'''' + QU'' + \lambda^2 U = 0 \tag{39a}$$

$$B.C. \qquad U''(0) = U'''(0) = 0 \tag{39b}$$

$$U''(1) = U'''(1) = 0 \tag{39c}$$

The situation becomes quite different when the compressive

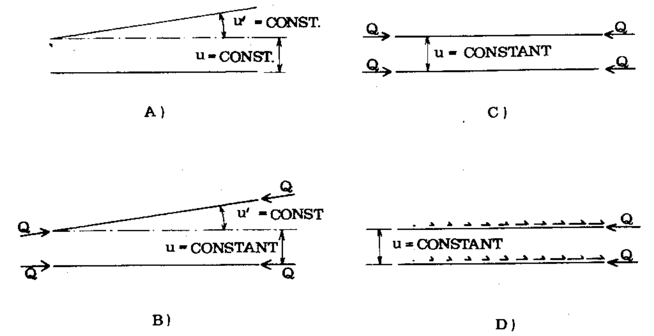

Fig. 2 Possible rigid body modes of some free-free beams. A) Free oscillation of a uniform beam. B) With follower force at both ends. C) Force with fixed direction at both ends. D) The model of a uniform missile.

force Q is to remain fixed in the direction of the undisturbed axis (Fig. 2c). The equations now become

$$D.E. \quad U'''' + QU'' + \lambda^2 U = 0 \tag{40a}$$

$$B.C. \quad U''(0) = U'''(0) + QU'(0) = 0 \tag{40b}$$

$$U''(1) = U'''(1) + QU'(1) = 0 \tag{40c}$$

In the system of Eq. (40), it can be readily shown that while rigid body translation (U= const, $\lambda^2 = 0$) is a solution, the rigid body rotation (U' = const) is not.

Similar situation exists for a uniform missile under a constant thrust at one end and with no directional control features (Fig. 2d). The system of equations is rewritten here.

$$D.E. \quad U'''' + (QXU')' + \lambda^2 U = 0 \tag{41a}$$

$$B.C. \quad U''(0) = U'''(0) = 0 \tag{41b}$$

$$U''(1) = U'''(1) = 0 \tag{41c}$$

Now it is amply clear that the only rigid body mode associated with Eqs. (41) is a translational motion. Rigid body rotation is again *not* a solution to the problem unless $Q = 0$, in which case, system (41) reduces to Eqs. (38).

Since Eqs. (38) can be considered as a special case of Eqs. (41) with $Q = 0$, it is reasonable to assume that there is a one-to-one correspondence between the solutions of the two systems. It is then a legitimate question to ask: What is happening to that particular eigenvalue λ^2 in Eq. (41) which corresponds to the rigid body rotation ($\lambda^2 = 0$) in Eqs. (38)? A numerical answer to this question is provided Sec.VI.

VI. Numerical Results

Since the final eigenvalue matrix is not symmetric for the problems considered here, a computer program for the eigenvalues of a general matrix must be used. In this study, two subroutines from the IBM Scientific Subroutines Package has been used. The first one, HSBG, transforms a given general matrix into an almost-upper-triangular, or the Hessenberg's, form. The second, ATEIG, extracts all the eigenvalues from a matrix in the Hessenberg's form. The calculations were performed in a computer system IBM360, Model 44, and the associated Operating System. Within the system, the double precision arithmetic carries sixteen significant digits and it is used throughout in the present study.

In a previous study on similar problems without any rigid body modes,[16] the convergence of the eigenvalues were found to be extremely good. Using six (6) elements, the eigenvalues of the first four lowest modes were obtained within one percent of the exact values. With one or two rigid body modes, the number of elements has to be increased to nine (9) in the present analysis.

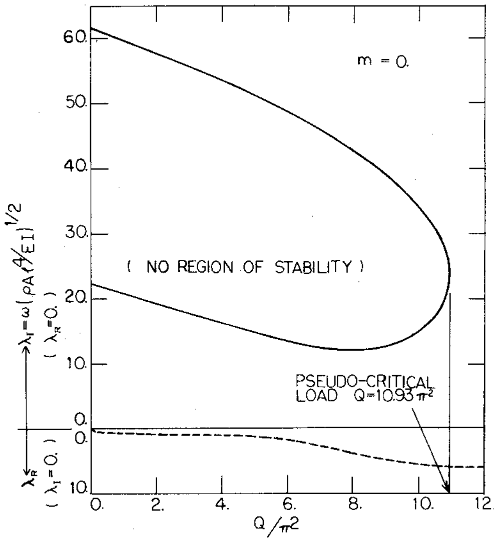

Fig. 3 Stability behavior of a uniform missile without thrust directional control; $m = 0$.

The numerical values of the first four lowest eigenvalues, including the zero mode, for a uniform missile without a concentrated mass and without directional control for the thrust are shown in Table 1 and plotted in Fig. 3. The zero eigenvalues are tabulated to emphasize the fact that they are the results of calculations and not by assumptions. For the special case of having no thrust, two zero eigenvalues are expected, and two obtained, representing two rigid body modes. With the application of any amount of Q, only one zero mode is obtained. As described Sec.V, only one rigid body mode, corresponding to translation, is expected and rigid body rotation is no longer a possible mode. Thus, the second lowest mode is expected to be nonzero. The numerical value of λ^2 turns out to be real and positive throughout the full range of Q, representing a divergence instability.

The third and fourth modes are the familiar vibratory modes which have already been reported by Beal,[1] Feodos'ev[2] and Matsumoto and Mote.[7] As the value of Q increases, these two branches eventually coalesce at $Q = 10.93\pi^2$. Beyond this point, λ becomes complex, representing flutter instability. However, this flutter instability is insignificant due to the second mode of divergence instability at any value of Q. It should be noted however that both Beal[1] and Matsumoto and Mote[7] have correctly discounted the significance of this flutter instability criterion, but only on incorrect grounds—that is, by the reasoning of rigid body rotations. Needless to say, a clear understanding of

Table 1 Numerical values of the four lowest eigenvalues for a uniform missile under constant thrust without directional control

Q/π^2	0.	1.	2.	3.	4.
λ_4	61.70	59.78	57.79	55.69	53.47
λ_3	22.37	20.95	19.49	17.98	16.45
λ_2	0.	(0.60)	(0.78)	(0.88)	(1.02)
λ_1	0.	0.	0.	0.	0.
Q/π^2	5.	6.	7.	8.	9.
λ_4	51.11	48.56	45.76	42.62	38.90
λ_3	14.94	13.57	12.50	12.08	12.70
λ_2	(1.29)	(1.82)	(2.64)	(3.69)	(4.72)
λ_1	0.	0.	0.	0.	0.

Table 2 Numerical values of the four lowest eigenvalues for a uniform missile with a concentrated mass at the tail end ($M=0.02$)

Q/π^2	0	1	2	3	4	5	6	7	8	9
λ_4	59.62	57.62	55.51	53.29	50.93	48.39	45.61	42.49	38.86	34.22
λ_3	21.58	20.04	18.42	16.73	14.95	13.11	11.28	9.90	10.29	12.94
λ_2	0.	(1.04)	(1.09)	(0.23)	1.64	2.73	3.76	4.33	2.90	(3.61)
λ_1	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

Table 3 Numerical values of the four lowest eigenvalues for a uniform missile with a concentrated mass at the tail end ($M=0.04$)

Q/π^2	0	1	2	3	4	5	6	7	8	9
λ_4	58.14	56.05	53.85	51.51	49.01	46.30	43.29	39.86	35.68	29.75
λ_3	20.94	19.29	17.52	15.60	13.47	10.85	—	—	—	—
λ_2	0.	(1.32)	(1.33)	0.86	2.72	4.82	—	—	—	—
λ_1	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

the stability behaviors of this simple problem is essential to the interpretation of data of more complicated cases.

The effect of a concentrated mass on the stability behavior of a uniform missile is discussed next. Let us consider a concentrated mass, two percent of the total mass of the beam, attached to the tail end (i.e., $m=0.02$, $x_m=1.0$). The eigenvalues of the four lowest modes are given in Table 2 and plotted in Fig. 4. In comparison with the case of having no concentrated mass, there is shown to be a stability zone between $Q=3.05\pi^2$ and $Q=8.41\pi^2$. Beyond this range at either end the structure is unstable due to divergence. There is also a pseudocritical load of flutter instability at $Q=10.05\pi^2$, at which the third and the fourth branches of the eigenvalue curve coalesce.

The stability behavior has again a characteristic change as the amount of the concentrates mass increases to 4% of the beam total mass ($m=0.04$, $x=1.0$). The eigenvalues of the four lowest modes are given in Table 3 and plotted in Fig. 5. Now the stability zone is between $Q=2.81\pi^2$ and $Q=5.59\pi^2$. Below $Q=2.81\pi^2$, the beam is still unstable due to convergence. However, the point $Q=5.59\pi^2$ marks the coalescence of the second branch and the third. Beyond it, flutter instability occurs.

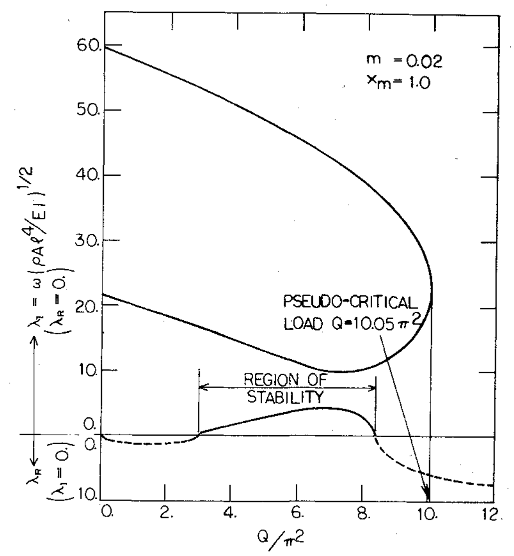

Fig. 4 Stability behavior of a uniform missile with a concentrated mass: $m=0.02, x_m=1.0$.

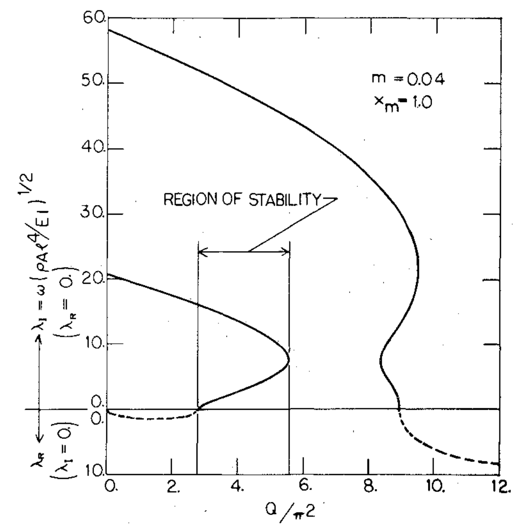

Fig. 5 Stability behavior of a uniform missile with a concentrated mass: $m=0.02, x_m 1.0$.

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